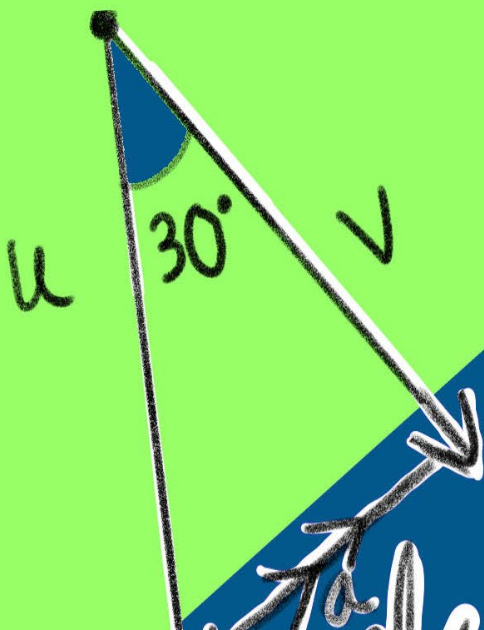
University of Southampton

BRIDGING GCSE Maths



with

To

A level PHYSICS

Contents

Preface

This booklet showcases the applications of *GCSE maths in A-level physics and mathematics* by exploring the mutual ground of mechanics. As an introduction to kinematics, it aims to stretch gifted math students into A level science, providing engaging activities for the eager student on holiday or in a summer workshop.

Prerequisites: Differentiation, trigonometry, quadratics and vectors. Calculators are allowed to get answers to 3s.f.

Answer Key: Available via the QR code below.

Acknowledgement:

This book was created as part of the outreach programme, and has made use of the extensive SMP archives at the University of Southampton.

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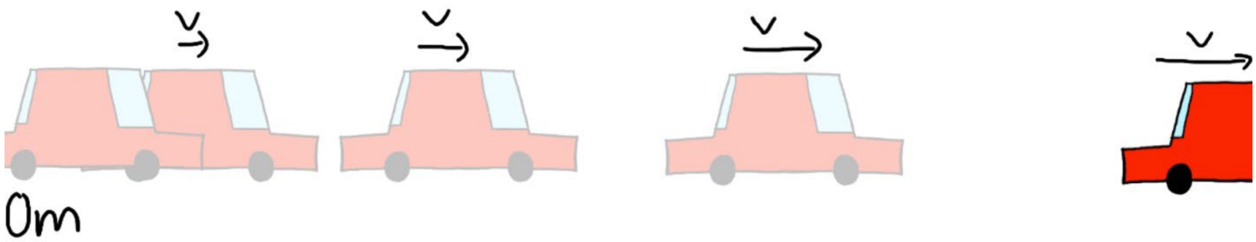
Rates of change

Introduction

Let's imagine a car at rest. It begins accelerating more and more and is soon out of sight. What would we do if we wanted to find how fast the car was moving, and accelerating two seconds into motion?



Say that the motion of the car is represented by $x = 2t^2$, where x is displacement in metres and t is in seconds for $0 \leq t \leq 20$
The velocity function (think of a tangent on a displacement-time graph) is found by

$$v = \frac{dx}{dt} = 4t$$

Differentiating the velocity function then gives acceleration.

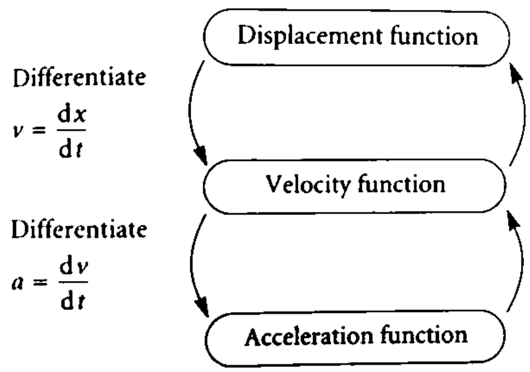
$$a = \frac{dv}{dt} = 4$$

Substituting $t = 2$ for the velocity equation yields

$$v = 4(2) = 8ms^{-1}$$

While $a = 4$, which stays constant regardless of the value of t

In GCSE, we are used to thinking of tangent drawing on graphs and differentiation as two different concepts. However, differentiation is the accurate way of getting gradient functions of displacement and velocity. This is because acceleration and velocity are the rates of change of velocity and displacement respectively. This is demonstrated by the following diagram:



Now try some similar questions:

1. A car takes 20 seconds to accelerate from rest to its terminal velocity, which is given by $v = \frac{1}{2}(40t - t^2)$ from $0 \leq t \leq 20$
Sketch an acceleration-time graph.

2. A 6 hour marathon runner's displacement can be modelled by the equation $x = 2t + 3t^2 - \frac{1}{3}t^3$ where x is in km and t in hours
How many hours into his journey was he running fastest in the forwards (positive) direction?

3. The motion of a particle to the right along a straight line is described by $x = t^3 - 6t^2 + 9t + 2$ where $t > 0$
a) When was the particle's velocity negative?
b) When was the particle's acceleration positive?
c) By drawing sign diagrams, at what two intervals of time was the particle slowing down?

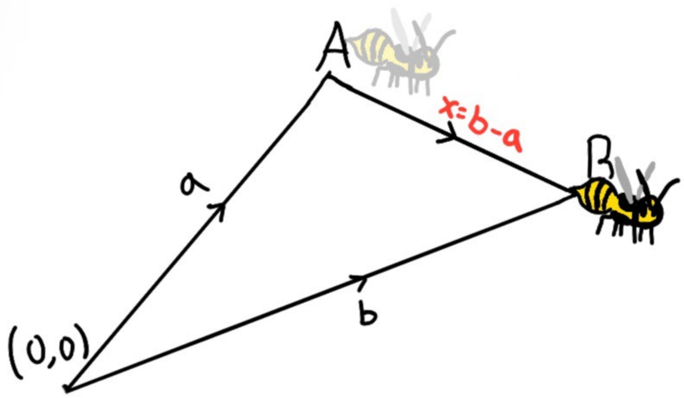
Extension Questions? The same principle is used for calculating all rates of change calculations.
4. If the volume of water in a vessel is modelled by $V = 1 - 3t - 3t^2$ where *V is* volume in m^3, after how many seconds was the vessel **losing** $9m^3$ of water per second?

Velocity and acceleration vectors

Introduction

Kinematics can also be navigated using vectors to model motion.

Suppose we have the ability to measure the position of a wasp at any instant. If at some instant the wasp is at A and a moment later it is at B, then its initial position vector a has changed by the displacement vector $AB = b - a$ to become position vector b during this interval of time.



Therefore, we can say that its

$$Average\ velocity\ from\ A\ to\ B = \frac{displacement\ AB}{tB - tA}$$

However, to find exact values of velocity we need to differentiate the position vector at A. This would yield the instantaneous velocity at point A, and the magnitude of this vector would be the scalar speed.

For example, let's say a ship moves so that its position vector at time t hours is

$$\begin{pmatrix} 2t \\ 6t - t^2 \end{pmatrix} = \begin{pmatrix} movement\ in\ x - axis \\ movement\ in\ y - axis \end{pmatrix}$$

What is the average velocity vector in the two hour interval $1 \le t \le 3$? Substituting the time values, we get position vectors:

$$t = 1, A = \binom{2}{5}$$
$$t = 3, B = \binom{6}{9}$$

And the displacement vector for the two hours is

$$AB = \binom{4}{4}$$

So dividing by 2 seconds, the average velocity vector is

$$\begin{pmatrix} \frac{4}{2} \\ \frac{4}{2} \end{pmatrix} = \binom{2}{2}$$

Then what are the instantaneous velocities at the two points?

$$p = \binom{2t}{6t - t^2} \rightarrow v = \frac{dp}{dt} = \binom{2}{6 - 2t}$$

Using substitution the instantaneous velocities are

t=1 $A = \binom{2}{4}$

t=3 $D = \binom{2}{0}$

Now try some similar questions: Unless otherwise specified, displacement is in metres, and time in seconds

1) A particle moves so that its position vector at time t is $\binom{t^2}{2t}$. Find its average velocity vector over the first three second interval.

2) A car moves so that its position vector at any time t is $\binom{4}{3t^2}$. Find instantaneous velocity at $t = 4$.

3) A particle moves so that after t seconds its velocity vector is $\binom{3\,cost}{-3sint}$ Show that its speed is always $3\ ms^{-1}$.

A natural follow on is that

$$average\ acceleration = \frac{vB - vA}{tB - tA}$$

The difference of two velocity vectors over time.

For example, going back to the example of the ship, for the same time interval $1 \leq t \leq 3$ we can find average acceleration using the instantaneous velocity vectors we found.

t=1 $A = \binom{2}{4}$

t=3 $D = \binom{2}{0}$

$$\bar{a} = \frac{\binom{2}{0} - \binom{2}{4}}{2}$$

$$= \frac{\binom{0}{-4}}{2} = \binom{0}{-2}$$

Or find the instantaneous acceleration at any point by differentiation of $v = \binom{2}{6-2t}$ which in this case is still

$$a = \frac{dv}{dt} = \binom{0}{-2}$$

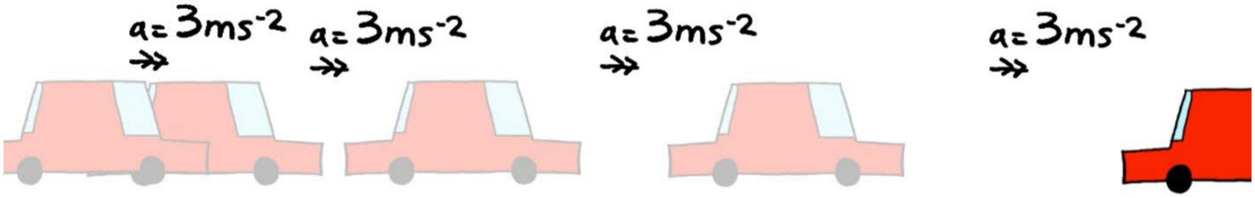
Extension Questions?

4) A car's velocity at $t = 2$ is $\binom{4}{5}$ and at $t = 6$ is $\binom{0}{17}$.
What is the average acceleration between these two points?

5) A ball moves so that its position vector at time t is $\binom{3t^2}{2t}$.
What is its instantaneous acceleration after 2 seconds?

Uniform Acceleration and displacement

Introduction



If the car starts with velocity *u* and accelerates steadily with acceleration vector *a,* then the velocity will change by set amount *a* every second.

$$v = u + at \ (1st\ equation\ of\ motion)$$

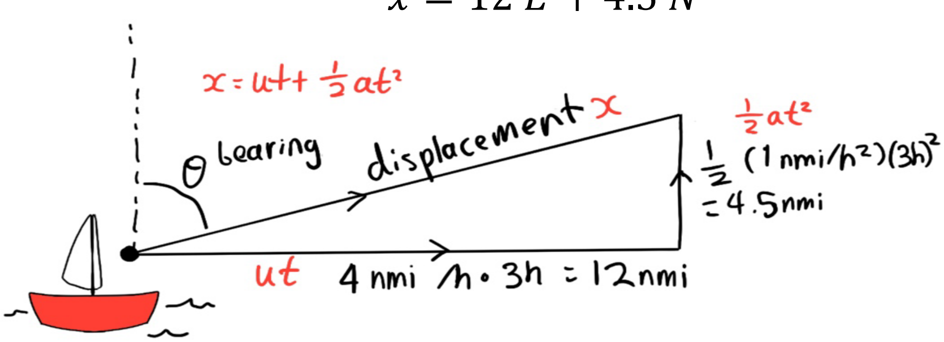
By simple integration of this velocity formula we also get the formula for displacement.

$$x = ut + \frac{1}{2}at^2 \ (2nd\ equation\ of\ motion)$$

Note that if the vectors ut and at^2 are in different directions, they can only be added directly if in column vector form.

Let's go through an example:
A boat travelling at 4 knots due east accelerates due north at 1 knot per hour, where $1knot = 1nautical\ mile\ per\ hour$. What is the magnitude and direction of displacement after 3 hours?

$$x = 4 \cdot 3\ E + \frac{1}{2}(1)(3)^2 N$$
$$x = 12\ E + 4.5\ N$$



$$x = \begin{pmatrix} 12 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 4.5 \end{pmatrix}$$

$$x = \begin{pmatrix} 12 \\ 4.5 \end{pmatrix} nmi$$

$$magnitude = \sqrt{12^2 + 4.5^2}$$

$$= 12.8\ nmi\ to\ 3.s.f.$$

$$\theta = \tan^{-1}\left(\frac{12}{4.5}\right)$$

$$= 069.4°$$

Now try some similar questions

1. A particle is moving with the velocity vector $\binom{50}{70}\ ms^{-1}$when it experiences a constant acceleration $\binom{4}{2}\ ms^{-2}$
What is its velocity and displacement after 5 seconds?

2. A rocket is observed to have a velocity of 200 ms^{-1}vertically upwards. 4 seconds later its velocity is 140 ms^{-1} at $45°$ to the horizontal. A further 5 seconds later its velocity is 200 ms^{-1} horizontally. Find the approximate uniform acceleration.

3. A spacecraft is in motion for half a minute, with an initial velocity of $600ms^{-1}\ NE$ and uniform acceleration of $10ms^{-2}N$
What is the magnitude and direction of its displacement?

Extension Questions?

4. A ball is thrown at 40 degrees to the horizontal with a speed of 20 ms^{-1}. At what height to the nearest metre would it hit a vertical wall which was 22m horizontal distance away?

This booklet is a GCSE student’s introduction to A level physics, with a focus on the widely applicable topic of kinematics. From looking at the difference of average and instantaneous acceleration, to applying Newton’s equations of motion, the booklet provides a bridge between GCSE maths to A level science.